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ON DENTABILITY AND THE BISHOP–PHELPS PROPERTY

BY

J. BOURGAIN

ABSTRACT

It is shown that for Banach spaces the Radon–Nikodym property and the Bishop–Phelps property are equivalent. Using similar techniques, we prove that if C is a bounded, closed and convex subset of a Banach space such that every nonempty subset of C is dentable, then the strongly exposing functionals of C form a dense G_δ-subset of the dual.

Preliminaries

All Banach spaces are assumed to be real Banach spaces. Let X be a Banach space with dual X'. For sets $A \subset X$, let $c(A)$ and $\bar{c}(A)$ denote the convex hull and closed convex hull, respectively. If $x \in X$ and $\varepsilon > 0$, then $B(x, \varepsilon) = \{y \in X; \| x - y \| < \varepsilon\}$.

A subset A of X is said to be dentable if for every $\varepsilon > 0$ there exists a point $x \in A$ such that $x \notin \bar{c}(A \setminus B(x, \varepsilon))$.

We will say that X is a dentable Banach space if every nonempty, bounded subset of X is dentable.

A Banach space X has the Radon–Nikodym property (RNP) provided for every measure space (Ω, Σ, μ) with $\mu(\Omega) < \infty$, and every μ-continuous measure $F: \Sigma \to X$ of finite variation, there exists a Bochner integrable function $f: \Omega \to X$ such that $F(E) = \int_E f \, d\mu$ for every $E \in \Sigma$.

The notion of dentability was introduced by M. A. Rieffel, who also proved that dentable Banach spaces have RNP [9]. The equivalence of the two properties was shown by W. J. Davis and R. R. Phelps [2] and simultaneously by R. E. Huff [4].

If Y is a Banach space, let $\mathscr{L}(X, Y)$ be the Banach space of all bounded linear operators from the Banach space X into the Banach space Y. The norm in $\mathscr{L}(X, Y)$ is the usual operator norm. Suppose $T \in \mathscr{L}(X, Y)$ and B a nonempty and bounded subset of X, then we take $N(T, B) = \sup\{\| Tx \|; x \in B\}$. A

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nonempty, bounded and closed subset B of X is said to possess the Bishop–Phelps property whenever given any Banach space Y and any operator $T \in \mathscr{L}(X, Y)$, there is an approximating sequence $(T_n)_n$ in $\mathscr{L}(X, Y)$, where each T_n achieves its max norm $N(T_n, B)$ on B. We will say that X has the Bishop–Phelps property if every bounded, closed and absolutely convex subset of X has the Bishop–Phelps property.

Proposition 1. *Let C be a nonempty, separable, bounded, closed and convex subset of X. If C has the Bishop–Phelps property, then C is dentable.*

Proof. Clearly we can assume that C is contained in the unit ball of X. Suppose that C is not dentable. Then, applying a result of Huff and Morris [5], there exists some $\varepsilon > 0$ so that $C = \bar{c}(C \setminus D)$, for each $D \subset X$ with an ε-convering. Let Z be the linear span of C and $(z_p)_p$ a dense sequence in Z. For $x \in X$ we define

$$||| x |||^2 = \| x \|^2 + \sum_{p=0}^{\infty} \frac{1}{2^p} \quad (\operatorname{dist}(x, \mathbf{R} z_p))^2.$$

Clearly $||| \quad |||$ is an equivalent norm on X. Consider the identity operator $I: X, \| \quad \| \to X, ||| \quad |||$. Since $C \subset X$ has the Bishop–Phelps property, there exists an operator $T: X, \| \quad \| \to X, ||| \quad |||$ achieving its max norm $N(T, C)$ on C and satisfying $\| I - T \| \leqq \varepsilon/4$. Take $x \in C$ with $x \neq 0$ and $||| Tx ||| = N(T, C)$. Since $x \in Z$, there exists some $q \in \mathbf{N}$ so that $\| x - z_q \| \leqq \varepsilon/4$. If $u_1, \cdots, u_d$ is a finite $\varepsilon/8$-net in $\mathbf{R} z_q \cap B(0, 1 + \varepsilon)$, then we find that

$$D_q = \left\{ y \in C; \operatorname{dist}(y, \mathbf{R} z_q) < \frac{7\varepsilon}{8} \right\} \subset \bigcup_{i=1}^{d} B(u_i, \varepsilon)$$

and therefore has an ε-covering. Hence $C = \bar{c}(C \setminus D_q)$ and therefore

$$2 ||| Tx ||| = \sup \{ ||| Tx + Ty |||; \ y \in C \setminus D_q \}.$$

Let $(y_n)_m$ be a sequence in $C \setminus D_q$ satisfying $2 ||| Tx ||| = \lim_{n \to \infty} ||| Tx + Ty_n |||$. From the definition of $||| \quad |||$ and the properties of the l^2-norm, it follows easily that $\operatorname{dist}(Tx, \mathbf{R} z_q) = \lim_{n \to \infty} \operatorname{dist}(Ty_n, \mathbf{R} z_q)$.

The fact that $\operatorname{dist}(Tx, \mathbf{R} z_q) \leqq \| Tx - x \| + \operatorname{dist}(x, \mathbf{R} z_q) \leqq \varepsilon/4 + \varepsilon/4 = \varepsilon/2$ and $\operatorname{dist}(Ty_n, \mathbf{R} z_q) \geqq \operatorname{dist}(y_n, \mathbf{R} z_q) - \| Ty_n - y_n \| \geqq 7\varepsilon/8 - \varepsilon/4 = 5\varepsilon/8$ yields the required contradiction.

Using a result of [5] and the fact that dentability is separably determined [7], we obtain immediately:

Corollary 2. *A Banach space with the Bishop–Phelps property has RNP.*

We will now pass to the proof of the converse.

LEMMA 3. *Let $(V_n)_n$ be a sequence of nonempty sets in X satisfying the following condition:*

There is an $\varepsilon > 0$ and a $\kappa > 0$ such that for each $z \in c(V_n)$ and each $p \in X$, $\operatorname{dist}(z, c(V_{n+1} \backslash B(p, \varepsilon))) < \kappa 2^{-n}$. *Then the set $A = \bigcap_{n=0}^{\infty} \overline{\bigcup_{j \geqq n} c(V_j)}$ is nonempty and not dentable.*

PROOF. First, we remark that $c(V_n) \subset A + B(0, \kappa 2^{-n+1})$. Indeed, if $z \in c(V_n)$, then there exists a sequence $(z_j)_{j \geqq n}$ such that $z_n = z$, $z_j \in c(V_j)$ and $\| z_j - z_{j+1} \| < \kappa 2^{-j}$.

Clearly $(z_j)_{j \geqq n}$ converges to some point $a \in A$ and furthermore $\| z - a \| < \kappa 2^{-n+1}$.

Now we show that if $x \in A$, then $x \in \bar{c}(A \backslash B(x, \varepsilon/2))$. Let $x \in A$ and let $0 < \tau < \varepsilon$. Take $n \in \mathbf{N}$ such that $\kappa 2^{-n+2} < \tau$. There is some $j \geqq n$ and some $z \in c(V_j)$ satisfying $\| x - z \| < \tau/2$.

Because $\operatorname{dist}(z, c(V_{j+1} \backslash B(x, \varepsilon))) < \kappa 2^{-j}$ and

$$V_{j+1} \backslash B(x, \varepsilon) \subset (A + B(0, \kappa 2^{-j})) \backslash B(x, \varepsilon) \subset \left(A \backslash B\left(x, \frac{\varepsilon}{2}\right)\right) + B(0, \kappa 2^{-j}),$$

it follows that $\operatorname{dist}(z, c(A \backslash B(x, \varepsilon/2))) < \kappa 2^{-j+1} < \tau/2$ and hence $\operatorname{dist}(x, c(A \backslash B(x, \varepsilon/2))) < \tau$. Since $\tau > 0$ can be taken arbitrarily small, $x \in \bar{c}(A \backslash B(x, \varepsilon/2))$. Thus A is not dentable.

LEMMA 4. *Let B be a nonempty, closed and absolutely convex subset of X, contained in the unit ball. Assume that every nonempty subset of B is dentable. Let Y be a Banach space. Let $\varepsilon > 0$ be given and define*

$$A_\varepsilon = \{T \in \mathcal{L}(X, Y);\ S(T, \eta) \subset B(p, \varepsilon) \cup B(-p, \varepsilon)$$
$$\text{for some} \quad \eta > 0 \quad \text{and} \quad p \in X\},$$

where $S(T, \eta) = \{x \in B;\ \| Tx \| \geqq N(T, B) - \eta\}$. Then A_ε is dense in $\mathcal{L}(X, Y)$. Moreover, if $\delta > 0$ and $S \in \mathcal{L}(X, Y)$, there is $T \in A_\varepsilon$ such that $\| S - T \| < \delta$ and $S - T$ is finite rank.

PROOF. Assume $\varepsilon > 0$, $0 < \delta < \frac{1}{2}$ and $S \in \mathcal{L}(X, Y)$. Clearly we can take $N(S, B) > 0$ and hence $N(S, B) = 1$. Now suppose that for every $T \in \mathcal{L}(X, Y)$ satisfying $\| S - T \| < \delta$ and $S - T$ finite rank, we have $T \notin A_\varepsilon$. For each $n \in \mathbf{N}$, let V_n be the set of those $x \in B$ for which there exists $T \in \mathcal{L}(X, Y)$ such that $\| Tx \| \geqq N(T, B) - 4^{-n}\delta^2$, $\| S - T \| \leqq \delta(1 - 2^{-n})$ and $S - T$ finite rank.

We claim that if $z \in V_n$ and $p \in X$, then $\operatorname{dist}(z, c(V_{n+1}\setminus B(p,\varepsilon))) < \kappa 2^{-n}$, where $\kappa = 2^6\delta$. Suppose $\operatorname{dist}(z, c(V_{n+1}\setminus B(p,\varepsilon))) \geqq \kappa 2^{-n}$. Since $D = V_{n+1}\setminus(B(p,\varepsilon)\cup B(-p,\varepsilon))$ is symmetric, there exists $f \in X'$ satisfying $\|f\| = 1$ and $f(z) \geqq \sup|f(D)| + \kappa 2^{-n}$.

Because $z \in V_n$, there is $T \in \mathscr{L}(X, Y)$ such that $\|Tz\| \geqq N(T,B) - 4^{-n}\delta^2$, $\|S - T\| \leqq \delta(1 - 2^{-n})$ and $S - T$ is finite rank. Thus $\frac{1}{2} < N(T,B) < 2$ and $\|Tz\| \geqq \frac{1}{4}$.

Let $\hat{T} \in \mathscr{L}(X, Y)$ be the operator given by

$$\hat{T}x = Tx + 2^{-n-2}\delta f(x)Tz.$$

Then $\|T - \hat{T}\| \leq 2^{-n-1}\delta$ and hence $\|S - \hat{T}\| \leqq \delta(1 - 2^{-n-1})$. Obviously $S - \hat{T}$ is still a finite rank operator. By hypothesis $\hat{T} \notin A_\varepsilon$ and thus there is some $x \in B$ with $x \notin B(p,\varepsilon) \cup B(-p,\varepsilon)$ and $\|\hat{T}x\| \geqq N(\hat{T}, B) - 4^{-n-1}\delta^2$. Clearly $x \in V_{n+1}$ and thus $x \in D$.

But $\|Tx\| + 2^{-n-2}\delta|f(x)|\|Tz\| \geqq \|\hat{T}z\| - 4^{-n-1}\delta^2$ implying

$$(1 + 2^{-n-2}\delta|f(x)|)\|Tz\| \geqq (1 + 2^{-n-2}\delta f(z))\|Tz\| - 2\, 4^{-n}\delta^2.$$

Therefore $|f(x)| \geqq f(z) - 2^{-n+5}\delta$, which contadicts $f(z) \geqq |f(x)| + \kappa 2^{-n}$. This proves the claim.

From the claim, it follows that the sequence $(V_n)_n$ of nonempty sets in X, $\varepsilon > 0$ and $\kappa > 0$ satisfy the condition of Lemma 3. Thus $A = \bigcap_{n=0}^{\infty} \overline{\bigcup_{j \geqq n} c(V_j)}$ is nonempty and not dentable. The fact that $A \subset B$ yields the final contradiction.

We introduce the following definition. Let B be a nonempty, bounded, closed and absolutely convex subset of X. Let Y be a Banach space and $T \in \mathscr{L}(X, Y)$. We will say that T is an absolutely strongly exposing operator for the set B if there exists some point x in B such that every sequence $(x_n)_n$ in B satisfying $N(T,B) = \lim_{n\to\infty}\|Tx_n\|$ has a subsequence coverging to x or to $-x$. Using a compactness argument, we observe that $T \in \mathscr{L}(X, Y)$ is an absolutely strongly exposing operator for the set B if and only if $T \in A_\varepsilon$, for every $\varepsilon > 0$, where A_ε is defined as in Lemma 4. Obviously such an operator T achieves its max norm $N(T, B)$ on B.

THEOREM 5. *Let B be a nonempty, bounded, closed and absolutely convex subset of X. Assume that every nonempty subset of B is dentable. Then for any Banach space Y the set A of the absolutely strongly exposing operators $T \in \mathscr{L}(X, Y)$ for the set B is a dense G_δ-subset of $\mathscr{L}(X, Y)$. In fact, if $S \in \mathscr{L}(X, Y)$ and $\delta > 0$, there is $T \in A$ such that $\|S - T\| \leqq \delta$ and $S - T$ is a compact operator.*

PROOF. Clearly B can be taken in the unit ball of X. For each $n \in \mathbf{N}^*$, we consider the subset $A_{1/n}$ of $\mathcal{L}(X, Y)$, which is open.

Indeed, assume $T \in A_{1/n}$ and let $S(T, \eta) \subset B(p, 1/n) \cup B(-p, 1/n)$ for some $\eta > 0$ and some $p \in X$. Then, if $U \in \mathcal{L}(X, Y)$ and $\| T - U \| < \eta/3$, we have $S(U, \eta/3) \subset S(T, \eta)$ and therefore $U \in A_{1/n}$.

Since $A = \bigcap_n A_{1/n}$, it follows from Lemma 4 that A is a dense G_δ in $\mathcal{L}(X, Y)$.

Now assume $S \in \mathcal{L}(X, Y)$ and $\delta > 0$. Let φ be the set of the compact operators C in $\mathcal{L}(X, Y)$ such that $\| C \| \leq \delta$. Then $S + \varphi$ is closed in $\mathcal{L}(X, Y)$ and again from Lemma 4 we obtain that $(S + \varphi) \cap A_{1/n}$ is dense in $S + \varphi$ for each $n \in \mathbf{N}^*$. Therefore A intersects $S + \varphi$ and every operator T in the intersection verifies the required properties.

COROLLARY 6. *Let B be a nonempty, bounded, closed and absolutely convex subset of X. Assume that every nonempty subset of B is dentable. Then for any Banach space Y the set of those operators $T \in \mathcal{L}(X, Y)$ which attain their* max norm $N(T, B)$ *on B is dense in $\mathcal{L}(X, Y)$. Hence B has the Bishop–Phelps property.*

Finally, Corollary 2 and Corollary 6 together give:

THEOREM 7. *A Banach spaceX has the Bishop–Phelps property if and only if it has RNP.*

We end with a result on the strongly exposing functionals of a convex set. Let C be a convex set in the Banach space X, then we will say that the point $x \in C$ is strongly exposed by $x^* \in X'$ if $x^*(x) = \max\ x^*(C)$ and if $\| x - x_n \| \to 0$ whenever each x_n is in C and $x^*(x_n) \to x^*(x)$. It was shown by R. R. Phelps [8] that if X is an RNP-space and C a bounded, closed and convex subset of X, then the functionals that strongly expose some point of C form a dense G_δ-subset of the dual X'.

The following theorem is stronger than Phelps' result and it also generalizes the well-known Troyanski–Lindenstrauss result on weakly compact convex sets [10].

THEOREM 8. *Let C be a nonempty, bounded, closed and convex subset of the Banach space X. Assume that every nonempty subset of C is dentable. Then the strongly exposing functionals of C form a dense G_δ-subset of X'.*

Using the same argument as in Theorem 5, Theorem 8 follows immediately from a slight modification of Lemma 4:

LEMMA 4'. *For* $\varepsilon > 0$, *let* $A_\varepsilon = \{x^* \in X', S(x^*, \eta) \subset B(p, \varepsilon)$ *for some* $\eta > 0$ *and* $p \in X\}$, *where* $S(x^*, \eta) = \{x \in C; x^*(x) \geqq \sup x^*(C) - \eta\}$. *Then* A_ε *is dense in* X'.

The proof of Lemma 4' is essentially the same as that of Lemma 4. We give an outline of it.

Assume C contained in the unit ball of $X, 0 < \delta < \frac{1}{2}, x^* \in X'$ and $\sup x^*(C) = 1$.

Now suppose that for every $y^* \in X'$ satisfying $\|x^* - y^*\| < \delta$ we have $y^* \notin A_\varepsilon$.

For each $n \in \mathbf{N}$, let V_n be the set of those $x \in C$ for which there exists $y^* \in X'$ such that $y^*(x) \geqq \sup y^*(C) - 4^{-n}\delta^2$ and $\|x^* - y^*\| \leqq \delta(1 - 2^{-n})$.

The only thing to show is that if $z \in V_n$ and $p \in X$, then $\operatorname{dist}(z, c(V_{n+1} \backslash B(p, \varepsilon))) < \kappa 2^{-n}$, where $\kappa = 2^6\delta$. If not, then we consider the set $D = V_{n+1} \backslash B(p, \varepsilon)$ and take $f \in X'$ satisfying $\|f\| = 1$ and $f(z) \geq \sup f(D) + \kappa 2^{-n}$. Because $z \in V_n$, there is $y^* \in X'$ such that $y^*(z) \geqq \sup y^*(C) - 4^{-n}\delta^2$ and $\|x^* - y^*\| \leqq \delta(1 - 2^{-n})$. Thus $\frac{1}{2} \leqq \sup y^*(C) \leqq 2$ and $y^*(z) \geqq \frac{1}{4}$.

Let $z^* \in X'$ be given by

$$z^*(x) = y^*(x) + 2^{-n-2}\delta f(x) y^*(z).$$

Then $\|z^* - y^*\| \leq 2^{-n-1}\delta$ and hence $\|x^* - z^*\| \leq \delta(1 - 2^{-n-1})$. By hypothesis $z^* \notin A_\varepsilon$ and thus there is some $x \in C$ with $x \notin B(p, \varepsilon)$ and $z^*(x) \geqq \sup z^*(C) - 4^{-n-1}\delta^2$.

Clearly $x \in V_{n+1}$ and thus $x \in D$. But

$$y^*(x) + 2^{-n-2}\delta f(x) y^*(z) \geqq z^*(z) - 4^{-n-1}\delta^2$$

implying $(1 + 2^{-n-2}\delta f(x)) y^*(z) \geqq (1 + 2^{-n-2}\delta f(z)) y^*(z) - 2\,4^{-n}\delta^2$. Therefore $f(x) \geqq f(z) - 2^{-n+5}\delta$, which contradicts $f(z) \geqq f(x) + \kappa 2^{-n}$.

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DEPARTMENT OF MATHEMATICS
VRIJA UNIVERSITEIT
BRUSSELS, BELGIUM